Midsemestral examination 2014
M.Math. II — Commutative algebra
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Throughout R stands for a commutative ring with $1 \neq 0$.

Q 1.
(a) Let $R[X]$ be the polynomial ring in one variable. Show that $R[X]$ is an integral domain if and only if R is an integral domain.
(b) Let R be a ring in which every element a has some power $a^{n(a)} = a$ with $n(a) > 1$. Show that all prime ideals of R must be maximal.
(c) Let $I_1, I_2, \ldots, I_n$ be ideals of R and p be a prime ideal of R such that $p = \bigcap_{i=1}^n I_i$. Show that $p = I_j$ for some j.

Q 2.
(a) Let M be a module over a local ring (R, m) and $x_1, x_2, \ldots, x_n \in M$. Show that if images of $x_1, x_2, \ldots, x_n$ in M/mM generate M/mM over R/m, then $x_1, x_2, \ldots, x_n$ generate M.
(b) Let M be an R-module and N, P be submodules of M. Show that $(N : P) = Ann(\frac{N+P}{N})$.
(c) Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Q 3.
(a) Determine whether $\mathbb{Z}/n\mathbb{Z}$ is flat as $\mathbb{Z}$-module for $n \geq 2$.
(b) Let M_α be a family of R-module, where α belongs to some indexing set Λ. Let $M = \oplus M_\alpha$. Show that M is flat R-module if and only if each M_α is flat.
(c) Let M and N be two R-modules. Show that if M and N are faithfully flat, then $M \otimes_R N$ is faithfully flat.

Q 4.
(a) Let A be a faithfully flat R-algebra and M be an R-module. If $A \otimes_R M$ is finitely generated over A , then show that M is finitely generated over R.
(b) Let P, Q be finitely generated R-modules with P projective and $J = rad(R)$. Let $\gamma \in Hom(Q, P)$ and *bar* denote deduction modulo J. If $\overline{\gamma} : \overline{Q} \longrightarrow \overline{P}$ is an isomorphism, show that γ is an isomorphism.

Q 5.
(a) Let M be an R-module and I be an ideal of R such that $M_m = 0$ for all maximal ideals m containing I. Show that $M = IM$.
(b) Let S be a multiplicatively closed set of R and I be an ideal of R. Show that $S^{-1}r(I) = r(S^{-1}I)$. Here $r(I)$ means radical of the ideal I.

Q 6.
(i) Let q be a p-primary ideal of R and $x \in R$. Show that if $x \notin q$, then $(q : x)$ is p-primary.
(ii) Let I be an ideal of R and let $I = \bigcap_{i=1}^{r} q_i$ be a minimal primary decomposition of I with $r(q_i) = p_i$. Show that for each i there exists $x_i \in R$ such that $(I : x_i)$ is p_i-primary.